

Quasi-tree expansion for the Bollobás–Riordan–Tutte polynomial

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Abstract

Oriented ribbon graphs (dessins d'enfant) are graphs embedded in oriented surfaces. The Bollobás–Riordan–Tutte polynomial is a three-variable polynomial that extends the Tutte polynomial to oriented ribbon graphs. A quasi-tree of a ribbon graph is a spanning subgraph with one face, which is described by an ordered chord diagram. We generalize the spanning tree expansion of the Tutte polynomial to a quasi-tree expansion of the Bollobás–Riordan–Tutte polynomial.

1 Introduction

An *oriented ribbon graph* is a multi-graph (loops and multiple edges allowed) that is embedded in an oriented surface. The embedding determines a cyclic order on the edges at every vertex. Other terms for oriented ribbon graphs include: combinatorial maps, fat graphs, cyclic graphs, graphs with rotation systems, and dessins d'enfant (see [1]). Bollobás and Riordan [1] extended the Tutte polynomial to an invariant of oriented ribbon graphs, now called the Bollobás–Riordan–Tutte polynomial. In [2], they generalized it to a four-variable invariant of non-orientable ribbon graphs. We only consider the orientable case, and henceforth all ribbon graphs will be oriented.

*Supported by National Science Foundation DMS-0455978.

†Supported by National Science Foundation DMS-0456227 and PSC-CUNY 60046-3637.

‡Supported by National Science Foundation DMS-0456275.

Originally, the Tutte polynomial was defined by a spanning-tree expansion using the concept of activity. In this paper, we generalize activity to ribbon graphs and give an expansion of the Bollobás–Riordan–Tutte polynomial over *quasi-trees*, which are spanning subgraphs with one face. In the genus zero case, we recover Tutte’s original expansion. In contrast to the expansion given in Section 6 of [2], our expansion has the advantage that quasi-trees provide fewer summands, and their weights are defined topologically.

Let $g(\mathbb{G})$ and $n(\mathbb{G})$ denote the genus and nullity of the ribbon graph \mathbb{G} . The Bollobás–Riordan–Tutte polynomial $C(\mathbb{G}) \in \mathbb{Z}[X, Y, Z]$ is recursively defined by $C(\mathbb{G}_1 \amalg \mathbb{G}_2) = C(\mathbb{G}_1) \cdot C(\mathbb{G}_2)$ and

$$C(\mathbb{G}) = \begin{cases} C(\mathbb{G} - e) + C(\mathbb{G}/e) & \text{if } e \text{ is neither a bridge nor a loop} \\ X \cdot C(D/e) & \text{if } e \text{ is a bridge} \\ \sum_{\mathbb{H} \subset \mathbb{G}} Y^{n(\mathbb{H})} Z^{g(\mathbb{H})} & \text{if } \mathbb{G} \text{ has one vertex} \end{cases}$$

Note that X is assigned to a bridge, and $1 + Y$ to a loop. For the Tutte polynomial, these variables are usually X and Y , respectively. If G is the underlying graph of a ribbon graph \mathbb{G} , then $C(\mathbb{G}; X, Y, 1) = T_G(X, 1 + Y)$.

Let $k(\mathbb{G})$ denote the number of components of a ribbon graph \mathbb{G} . The Bollobás–Riordan–Tutte polynomial has the following spanning subgraph expansion:

$$C(\mathbb{G}) = \sum_{\mathbb{H} \subset \mathbb{G}} (X - 1)^{k(\mathbb{H}) - k(\mathbb{G})} Y^{n(\mathbb{H})} Z^{g(\mathbb{H})} \quad (1)$$

We will use (1) to prove a quasi-tree expansion for the Bollobás–Riordan–Tutte polynomial. Recall the spanning tree expansion for the Tutte polynomial [4]: For any connected graph G with an order on the edges,

$$T_G(x, y) = \sum_{T \subset G} x^{i(T)} y^{j(T)}$$

where $i(T)$ is the number of internally active edges and $j(T)$ is the number of externally active edges for a given spanning tree T of G .

Fix a total order on the edges of a connected ribbon graph \mathbb{G} . In Proposition 1 below, we show that a quasi-tree \mathbb{Q} is given by a unique ordered chord diagram, which we use to define activities (*live* or *dead*) for edges of \mathbb{G} with respect to \mathbb{Q} . Let $\mathcal{D}(\mathbb{Q})$ be the subgraph of dead edges in \mathbb{Q} (*internally dead edges*). Let $\mathcal{I}(\mathbb{Q})$ be the set of live edges in \mathbb{Q} (*internally live edges*). Let $\mathcal{E}(\mathbb{Q})$ be the set of live edges in $\mathbb{G} - \mathbb{Q}$ (*externally live edges*).

For a given quasi-tree let $G_{\mathbb{Q}}$ denote the graph whose vertices are the components of $\mathcal{D}(\mathbb{Q})$ and whose edges are the internally live edges of \mathbb{Q} . Let $T_{G_{\mathbb{Q}}}(x, y)$ denote the Tutte polynomial of $G_{\mathbb{Q}}$. Our main result is the following:

Theorem 1 *Fix a total order on the edge set of a connected ribbon graph \mathbb{G} . With the preceding notation, there is a quasi-tree expansion for the Bollobás–Riordan–Tutte polynomial given by:*

$$C(\mathbb{G}) = \sum_{\mathbb{Q} \subset \mathbb{G}} Y^{n(\mathcal{D}(\mathbb{Q}))} Z^{g(\mathcal{D}(\mathbb{Q}))} (1+Y)^{|\mathcal{E}(\mathbb{Q})|} T_{G_{\mathbb{Q}}}(X, 1+YZ)$$

Let $\mathcal{B}(\mathbb{Q})$ and $\mathcal{N}(\mathbb{Q})$ be the set of internally live edges of \mathbb{Q} that are, respectively, bridges and edges that join the same component of $\mathcal{D}(\mathbb{Q})$. Thus, $G_{\mathbb{Q}}$ has $|\mathcal{B}|$ bridges and $|\mathcal{N}|$ loops, which contribute factors $X^{|\mathcal{B}|}$ and $(1+YZ)^{|\mathcal{N}|}$ to $T_{G_{\mathbb{Q}}}(X, 1+YZ)$ in Theorem 1. In the case when \mathbb{G} has a single vertex, there are only loops, so we have the following simplification:

Corollary 2 *Fix a total ordering of the edges of a ribbon graph \mathbb{G} with one vertex.*

$$C(\mathbb{G}) = \sum_{\mathbb{Q} \subset \mathbb{G}} Y^{n(\mathcal{D}(\mathbb{Q}))} Z^{g(\mathcal{D}(\mathbb{Q}))} (1+Y)^{|\mathcal{E}(\mathbb{Q})|} (1+YZ)^{|\mathcal{I}(\mathbb{Q})|}$$

In the case when \mathbb{G} is planar, i.e. $g(\mathbb{G}) = 0$, the quasi-trees are spanning trees and $G_{\mathbb{Q}}$ is a tree with $|\mathcal{I}(\mathbb{Q})|$ edges. After substituting $Y = y - 1$ and $Z = 1$ in $C(\mathbb{G})$, we recover Tutte’s original spanning tree expansion from Theorem 1.

2 Activities with respect to a quasi-tree

Let \mathbb{G} be a connected ribbon graph given by permutations $(\sigma_0, \sigma_1, \sigma_2)$ of $\{1, \dots, 2n\}$, such that σ_1 is a fixed-point free involution and σ_2 is defined by $\sigma_0 \sigma_1 \sigma_2 = 1$ (then $\sigma_2 = \sigma_1 \circ \sigma_0^{-1}$). The orbits of σ_0 form the vertex set, the orbits of σ_1 form the edge set, and the orbits of σ_2 form the face set. Let $v(\mathbb{G})$, $e(\mathbb{G})$ and $f(\mathbb{G})$ be the numbers of vertices, edges and faces of \mathbb{G} . The preceding data determine an embedding of the ribbon graph on a closed orientable surface, denoted $S(\mathbb{G})$, as a cell complex. The set $\{1, \dots, 2n\}$ can be identified with the directed edges (or half-edges). A subgraph $\mathbb{H} \subset \mathbb{G}$ is called a spanning subgraph if $V(\mathbb{H}) = V(\mathbb{G})$. Following Definition 3.1 of [3], we have:

Definition 1 *A quasi-tree \mathbb{Q} is a spanning subgraph of \mathbb{G} with $f(\mathbb{Q}) = 1$.*

For any spanning subgraph \mathbb{H} , a regular neighborhood of \mathbb{H} can be constructed on the surface $S(\mathbb{G})$ by gluing 2-discs at each vertex and rectangular bands whose midlines are the edges of \mathbb{H} . Let $\gamma_{\mathbb{H}}$ be the union of simple closed curves that bound such a regular neighborhood of \mathbb{H} .

Let an *ordered chord diagram* denote a circle marked with $\{1, \dots, 2n\}$ in some order, and chords joining all pairs $\{i, \sigma_1(i)\}$.

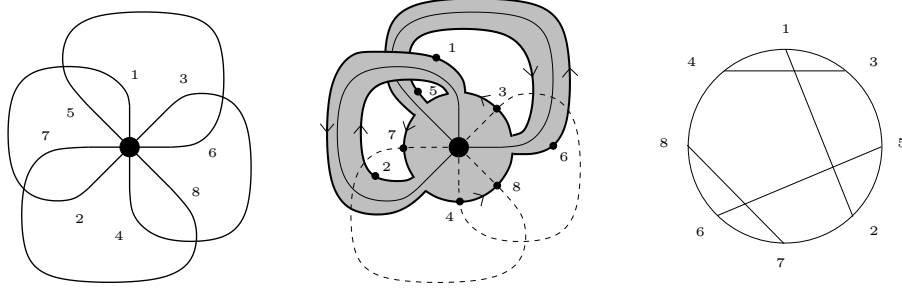


Figure 1: Ribbon Graph \mathbb{G} , quasi-tree $\mathbb{Q} = (12)(56)$ with curve $\gamma_{\mathbb{Q}}$, chord diagram $C_{\mathbb{Q}}$

Proposition 1 *Let \mathbb{G} be a connected ribbon graph. Every quasi-tree \mathbb{Q} of \mathbb{G} corresponds to the ordered chord diagram $C_{\mathbb{Q}}$ with consecutive markings in the positive direction given by the permutation:*

$$\sigma(i) = \begin{cases} \sigma_0(i) & i \notin \mathbb{Q} \\ \sigma_2^{-1}(i) & i \in \mathbb{Q} \end{cases}$$

Proof: Since \mathbb{Q} is a quasi-tree, $\gamma_{\mathbb{Q}}$ is one simple closed curve. If we choose an orientation on $S(\mathbb{G})$, we can traverse $\gamma_{\mathbb{Q}}$ along successive boundaries of bands and vertex discs, such that we always travel around the boundary of each disc in a positive direction (i.e., the disc is on the left). If a half-edge is not in \mathbb{Q} , $\gamma_{\mathbb{Q}}$ will pass across it travelling along the boundary of a vertex disc to the next band. If a half-edge is in \mathbb{Q} , $\gamma_{\mathbb{Q}}$ traverses along one of the edges of its band. On $\gamma_{\mathbb{Q}}$, we mark a half-edge not in \mathbb{Q} when $\gamma_{\mathbb{Q}}$ passes across it along the boundary of the vertex disc, and we mark a half-edge in \mathbb{Q} when we traverse an edge of a band in the direction of the half-edge. If the half-edge i is not in \mathbb{Q} , travelling along the boundary of a vertex disc, the next half-edge is given by σ_0 . If the half-edge i is in \mathbb{Q} , traversing the edge of its band to the vertex disc and then along the boundary of that disc, the next half-edge is given by $\sigma_0\sigma_1 = \sigma_2^{-1}$. For example, see Figure 1.

As \mathbb{Q} is a quasi-tree, each of its half-edges must be in the orbit of its single face, while the complementary set of half-edges are met along the boundaries of the vertex discs. Since we mark all half-edges traversing $\gamma_{\mathbb{Q}}$, the chord diagram $C_{\mathbb{Q}}$ parametrizes $\gamma_{\mathbb{Q}}$. ■

Definition 2 *Using $\min(i, \sigma_1(i))$, there is an induced total order on the chords of $C_{\mathbb{Q}}$. A chord is live if it does not intersect lower-ordered chords, and otherwise it is dead. For any quasi-tree \mathbb{Q} , an edge e is live or dead when the corresponding chord of $C_{\mathbb{Q}}$ is live or dead; and e is internal or external, according to $e \in \mathbb{Q}$ or $e \in \mathbb{G} - \mathbb{Q}$, respectively.*

In Figure 1, we show $C_{\mathbb{Q}}$ such that the only edge live with respect to \mathbb{Q} is (12) , which is internally live.

Note that if $g(\mathbb{G}) = 0$ then the underlying graph is planar, the only quasi-trees are spanning trees, and this definition reduces to activities in the sense of Tutte.

3 Binary tree of spanning subgraphs

The spanning subgraphs of a given ribbon graph \mathbb{G} form a poset (of states) \mathcal{P} isomorphic to the boolean lattice, $\{0, 1\}^{E(\mathbb{G})}$ of subsets of the set of edges. The partial order is given by $e = (e_i) \prec f = (f_i)$ provided $e_i < f_i$ for all i . In this section, we define a binary tree \mathcal{T} , which is similar to the skein resolution tree for diagrams widely used in knot theory. By the construction below, the leaves of \mathcal{T} correspond exactly to quasi-trees of \mathbb{G} .

A *resolution* of \mathbb{G} is a function $s : E(\mathbb{G}) \rightarrow \{0, 1\}$, which determines a spanning subgraph $\mathbb{H}_s = \{e \in \mathbb{G} \mid s(e) = 1\}$. Let $\rho : E(\mathbb{G}) \rightarrow \{0, 1, *\}$ be a *partial resolution* of \mathbb{G} , with edges called *unresolved* if they are assigned $*$. Let $\mathbb{H}_\rho = \{e \in \mathbb{G} \mid \rho(e) = 1\}$. A partial resolution determines an interval in the poset, $[\rho] = \{s \mid s(e_i) = \rho(e_i) \text{ if } \rho(e_i) \in \{0, 1\}\} = [\rho \wedge 0, \rho \wedge 1]$, the interval between $\rho \wedge 0$ with all unresolved edges of ρ set to zero, and $\rho \wedge 1$ with all unresolved edges of ρ set to one.

Definition 3 *If e is an unresolved edge in a partial resolution ρ , let ρ_0^e, ρ_1^e be partial resolutions obtained from ρ by resolving e to be 0 and 1, respectively. Then e is called nugatory if one of the intervals $[\rho_0^e]$ or $[\rho_1^e]$ contains no quasitrees.*

For example, when $g(\mathbb{G}) = 0$, an edge e is nugatory in ρ if adding it completes a cycle in ρ_1^e , or ρ_0^e is disconnected and no unresolved edges can connect it back.

Let $\gamma_\rho = \gamma_{\mathbb{H}_\rho}$, which was defined previously as the boundary of a certain regular neighborhood of \mathbb{H}_ρ , and let $|\gamma_\rho|$ denote the number of its components. Then $|\gamma_\rho| = f(\mathbb{H}_\rho)$, the number of faces on $S(\mathbb{H}_\rho)$, the associated surface for \mathbb{H}_ρ .

Lemma 1 *Given a partial resolution ρ , let $e \in \mathbb{G}$ be an unresolved edge. For $i \in \{0, 1\}$, let $\rho_i = \rho_i^e$, and let*

$$\gamma_i(\rho, e) = \gamma_{\rho_i} \cup \text{Int}(\rho_i^{-1}(*))$$

The edge e is nugatory if and only if either $\gamma_0(\rho, e)$ or $\gamma_1(\rho, e)$ is disconnected on $S(\mathbb{G})$. If $\gamma_0(\rho, e)$ is disconnected then $|\gamma_{\rho_1}| = |\gamma_\rho| - 1$ and $|\gamma_{\rho_0}| = |\gamma_\rho|$. If $\gamma_1(\rho, e)$ is disconnected then $|\gamma_{\rho_0}| = |\gamma_\rho|$ and $|\gamma_{\rho_1}| = |\gamma_\rho| + 1$.

Proof: Given a partial resolution ρ , we call \mathbb{H}_ρ *split* if $f(\mathbb{H}_\rho \cup U) > 1$ for all subsets U of unresolved edges. An unresolved edge e of ρ is nugatory if and only if either \mathbb{H}_{ρ_0} or \mathbb{H}_{ρ_1} is split. Since $f(\mathbb{H}_\rho) = |\gamma_\rho|$, \mathbb{H}_{ρ_0} or \mathbb{H}_{ρ_1} is split if and only if deleting e or cutting along e , respectively, disconnects $\gamma_\rho \cup \text{Int}(\rho^{-1}(*))$. Hence, for $i \in \{0, 1\}$, \mathbb{H}_{ρ_i} is split if and only if $\gamma_i(\rho, e)$ is disconnected.

If $\gamma_0(\rho, e)$ is disconnected then e is the only edge connecting two components of γ_ρ . Hence, these two components are connected in γ_{ρ_1} . This gives $|\gamma_{\rho_1}| = |\gamma_\rho| - 1$ and $|\gamma_{\rho_0}| = |\gamma_\rho|$. On the other hand, if $\gamma_1(\rho, e)$ is disconnected, then e intersects a component of γ_ρ twice without linking any other unresolved edge, so this component becomes

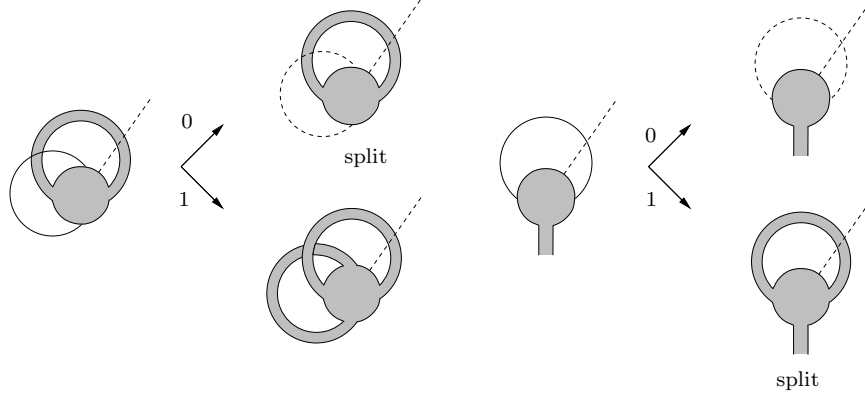


Figure 2: Two possibilities for a nugatory edge


disconnected in γ_{ρ_1} . This gives $|\gamma_{\rho_0}| = |\gamma_\rho|$ and $|\gamma_{\rho_1}| = |\gamma_\rho| + 1$. Here is a sketch of the two cases:  ■

Figure 2 shows two possibilities for a nugatory edge.

Lemma 2 *For any connected ribbon graph \mathbb{G} with ordered edges, there exists a rooted binary tree \mathcal{T} of \mathbb{G} , for which each node corresponds to a partial resolution ρ of \mathbb{G} . The leaves of \mathcal{T} are spanning subgraphs, all of whose unresolved edges are nugatory. Moreover, the partial resolution of a leaf can be resolved uniquely to give a quasi-tree.*

Proof: Let the root of \mathcal{T} be the totally unresolved partial resolution, $\rho(e) = *$ for all e . We resolve edges by changing $*$ to 0 or 1 in the reverse order (starting with highest ordered edge). If an edge is nugatory, the edge is left unresolved, and we proceed to the next edge. For a given node ρ in \mathcal{T} , if e is not nugatory then the left child is ρ_0^e and the right child is ρ_1^e . We terminate this process at a leaf when all subsequent edges are nugatory, and return as far back up \mathcal{T} as necessary to a node with a non-nugatory edge still left to be resolved. Therefore, the leaves of \mathcal{T} are spanning subgraphs of \mathbb{G} all of whose unresolved edges are nugatory.

By construction, for a leaf ρ of \mathcal{T} , \mathbb{H}_ρ is not split, so there exists a resolution $s \in [\rho]$ such that $f(\mathbb{H}_s) = |\gamma_{\mathbb{H}_s}| = 1$. In particular, since all unresolved edges are nugatory, by Lemma 1, there is a unique resolution $s \in [\rho]$ such that $|\gamma_{\mathbb{H}_s}|$ is minimized. Including nugatory edges e for which $\gamma_1(\rho, e)$ is connected, and excluding nugatory edges e for which $\gamma_1(\rho, e)$ is disconnected, $|\gamma_{\mathbb{H}_s}| = 1$. Hence, \mathbb{H}_s is a quasi-tree. ■

Lemma 3 *Let \mathcal{T} be the rooted binary tree obtained in Lemma 2. Let ρ be a leaf of \mathcal{T} , and let $\mathbb{Q} \in [\rho]$ be the corresponding quasi-tree. If $\rho(e) = *$ then e is live with respect to \mathbb{Q} , and otherwise e is dead with respect to \mathbb{Q} .*

Proof: Let e_i and e_j be unresolved edges of ρ , which are nugatory by Lemma 2. Moreover, by Lemma 2, the resolution $s \in [\rho]$ such that $\mathbb{H}_s = \mathbb{Q}$ is unique. If e_i and e_j link each other on $\gamma_{\mathbb{Q}}$, then the resolution s' with both $s(e_i)$ and $s(e_j)$ changed satisfies $1 = |\gamma_s| = |\gamma_{s'}|$ and hence $\mathbb{H}_{s'}$ is a quasitree for a second resolution $s' \in [\rho]$, which is a contradiction. Thus, unresolved edges can only link resolved edges.

Suppose e_i is unresolved and links a resolved edge e_j with $i > j$. By Lemma 2, there exists a unique closest parent $\tilde{\rho}$ of ρ in \mathcal{T} , such that e_j is a non-nugatory unresolved edge in $\tilde{\rho}$. Since edges are resolved in the reverse order, e_i is nugatory in $\tilde{\rho}$. As e_i links e_j , $\gamma_0(\tilde{\rho}, e_i)$ and $\gamma_1(\tilde{\rho}, e_i)$ are connected, which contradicts Lemma 1. Thus, if e_i and e_j are linked then $i < j$, so e_i is live.

Now, let e_i be a resolved edge of ρ . By Lemma 2, there exists a unique closest parent $\tilde{\rho}$ of ρ in \mathcal{T} , such that e_i is a non-nugatory unresolved edge in $\tilde{\rho}$. By Lemma 1, $\gamma_0(\tilde{\rho}, e_i)$ and $\gamma_1(\tilde{\rho}, e_i)$ are both connected. Hence, there exists e_j , which is unresolved in $\tilde{\rho}$, such that e_i and e_j are linked. If e_j is resolved after e_i in \mathcal{T} , $i > j$, so e_i is dead. If e_j is left unresolved in \mathcal{T} , it is live by the argument above, so e_i is dead. ■

To summarize, we have proved the following theorem:

Theorem 3 *For any connected ribbon graph \mathbb{G} with ordered edges, there exists a rooted binary tree \mathcal{T} of \mathbb{G} whose nodes are partial resolutions ρ of \mathbb{G} , and whose leaves correspond to quasi-trees \mathbb{Q} of \mathbb{G} . If the leaf ρ corresponds to \mathbb{Q} , then its unresolved edges are nugatory, and they can be uniquely resolved to obtain \mathbb{Q} . In \mathbb{G} , these are exactly the live edges with respect to \mathbb{Q} .*

4 Proof of Theorem 1

Let $\mathbb{H} \subset \mathbb{G}$ be a spanning subgraph. Let $n(\mathbb{H})$, $g(\mathbb{H})$ and $k(\mathbb{H})$ denote the nullity, genus and number of components of \mathbb{H} , respectively. Since $v(\mathbb{H}) = v(\mathbb{G})$,

$$n(\mathbb{H}) = k(\mathbb{H}) - v(\mathbb{G}) + e(\mathbb{H}), \quad g(\mathbb{H}) = \frac{2k(\mathbb{H}) - v(\mathbb{G}) + e(\mathbb{H}) - f(\mathbb{H})}{2}$$

Let \mathbb{Q} be a quasi-tree of \mathbb{G} . Let $\mathcal{I} = \mathcal{I}(\mathbb{Q})$ and $\mathcal{E} = \mathcal{E}(\mathbb{Q})$ be the internally and externally live edges with respect to \mathbb{Q} . Let $\mathcal{D} = \mathcal{D}(\mathbb{Q})$, the subgraph of dead edges in \mathbb{Q} . Let $n_0 = n(\mathcal{D})$ and $g_0 = g(\mathcal{D})$.

By Theorem 3, there is a unique partial resolution ρ of \mathbb{G} that is a leaf of \mathcal{T} , for which $\mathbb{Q} \in [\rho]$, and all resolutions \mathbb{H}_s for $s \in [\rho]$ are of the form $\mathcal{D} \cup S$ where $S \subset \mathcal{I} \cup \mathcal{E}$. All resolutions \mathbb{H}_s are leaves of the state poset \mathcal{P} , so the sum in (1) is the state sum for \mathcal{P} . The sum in Theorem 1 is the state sum for \mathcal{T} . Below, we prove that these two state sums are equal.

Let $S = S_1 \cup S_2$, where $S_1 \subset \mathcal{I}$ and $S_2 \subset \mathcal{E}$. By Lemma 4, the contribution from $[\rho]$ to

the sum in equation (1) is

$$\begin{aligned}
& \sum_{S \subset \mathcal{I} \cup \mathcal{E}} (X-1)^{k(\mathcal{D} \cup S)-1} Y^{n(\mathcal{D} \cup S)} Z^{g(\mathcal{D} \cup S)} \\
&= \sum_{S_2 \subset \mathcal{E}} Y^{|S_2|} \sum_{S_1 \subset \mathcal{I}} (X-1)^{k(\mathcal{D} \cup S_1)-1} Y^{n(\mathcal{D} \cup S_1)} Z^{g(\mathcal{D} \cup S_1)} \\
&= (1+Y)^{|\mathcal{E}|} \sum_{S_1 \subset \mathcal{I}} (X-1)^{k(\mathcal{D} \cup S_1)-1} Y^{n(\mathcal{D} \cup S_1)} Z^{g(\mathcal{D} \cup S_1)}
\end{aligned}$$

Let $G_{\mathbb{Q}}$ denote the graph whose vertices are the components of \mathcal{D} and whose edges are the edges in \mathcal{I} . \mathbb{Q} is a connected subgraph of \mathbb{G} , so $G_{\mathbb{Q}}$ is a connected graph, hence $k(G_{\mathbb{Q}}) = 1$. The subgraphs $\{\mathcal{D} \cup S_1 \mid S_1 \subset \mathcal{I}\}$ are in one-one correspondence with spanning subgraphs $W \subset G_{\mathbb{Q}}$. By Lemma 5 below,

$$\begin{aligned}
& \sum_{S_1 \subset \mathcal{I}} (X-1)^{k(\mathcal{D} \cup S_1)-1} Y^{n(\mathcal{D} \cup S_1)} Z^{g(\mathcal{D} \cup S_1)} \\
&= \sum_{W \subset G_{\mathbb{Q}}} (X-1)^{k(W)-1} Y^{n(\mathcal{D})+n(W)} Z^{g(\mathcal{D})+n(W)} \\
&= Y^{n_0} Z^{g_0} \sum_{W \subset G_{\mathbb{Q}}} (X-1)^{k(W)-k(G_{\mathbb{Q}})} (YZ)^{n(W)} \\
&= Y^{n_0} Z^{g_0} T_{G_{\mathbb{Q}}}(X, 1+YZ)
\end{aligned}$$

The last step is obtained from the spanning subgraph expansion of the Tutte polynomial:

$$T_G(x, y) = \sum_{W \subset G} (x-1)^{k(W)-k(G)} (y-1)^{n(W)}$$

with $x = X$ and $y = 1 + YZ$. ■

Lemma 4 *For a quasi-tree \mathbb{Q} of \mathbb{G} , let $S = S_1 \cup S_2$, where $S_1 \subset \mathcal{I}(\mathbb{Q})$ and $S_2 \subset \mathcal{E}(\mathbb{Q})$.*

1. $k(\mathcal{D}(\mathbb{Q}) \cup S) = k(\mathcal{D}(\mathbb{Q}) \cup S_1)$
2. $n(\mathcal{D}(\mathbb{Q}) \cup S) = n(\mathcal{D}(\mathbb{Q}) \cup S_1) + |S_2|$
3. $g(\mathcal{D}(\mathbb{Q}) \cup S) = g(\mathcal{D}(\mathbb{Q}) \cup S_1)$

Proof: Let $e \in \mathcal{E}(\mathbb{Q})$. By Theorem 3, \mathbb{Q} corresponds to ρ such that e is nugatory, and by Lemma 1, $\gamma_0(\rho, e)$ is connected. Hence, e intersects only one component of $\gamma_{\mathcal{D}}$. Thus, $k(\mathcal{D} \cup e) = k(\mathcal{D})$, which proves part 1.

$$\begin{aligned}
n(\mathcal{D} \cup S) &= k(\mathcal{D} \cup S) - v(\mathbb{G}) + e(\mathcal{D} \cup S) \\
&= k(\mathcal{D} \cup S_1) - v(\mathbb{G}) + e(\mathcal{D} \cup S_1) + |S_2| \\
&= n(\mathcal{D} \cup S_1) + |S_2|
\end{aligned}$$

Since $f(\mathbb{H}) = |\gamma_{\mathbb{H}}|$, by Lemma 1, $f(\mathcal{D} \cup e) = f(\mathcal{D}) + 1$, hence

$$\begin{aligned}
2g(\mathcal{D} \cup S) &= 2k(\mathcal{D} \cup S) - v(\mathbb{G}) + e(\mathcal{D} \cup S) - f(\mathcal{D} \cup S) \\
&= 2k(\mathcal{D} \cup S_1) - v(\mathbb{G}) + (e(\mathcal{D} \cup S_1) + |S_2|) - (f(\mathcal{D} \cup S_1) + |S_2|) \\
&= 2k(\mathcal{D} \cup S_1) - v(\mathbb{G}) + e(\mathcal{D} \cup S_1) - f(\mathcal{D} \cup S_1) \\
&= 2g(\mathcal{D} \cup S_1)
\end{aligned}$$

■

Lemma 5 *For a quasi-tree \mathbb{Q} of \mathbb{G} , let $S_1 \subset \mathcal{I}(\mathbb{Q})$. Let W be the spanning subgraph of $G_{\mathbb{Q}}$ whose edges are the edges in S_1 .*

1. $n(\mathcal{D}(\mathbb{Q}) \cup S_1) = n(\mathcal{D}(\mathbb{Q})) + n(W)$
2. $g(\mathcal{D}(\mathbb{Q}) \cup S) = g(\mathcal{D}(\mathbb{Q})) + n(W)$

Proof: For spanning subgraph W of $G_{\mathbb{Q}}$, $k(W) = k(\mathcal{D} \cup S_1)$. Hence,

$$\begin{aligned}
n(W) &= k(W) - v(G_{\mathbb{Q}}) + e(W) = k(\mathcal{D} \cup S_1) - k(\mathcal{D}) + |S_1| \\
n(\mathcal{D} \cup S_1) &= k(\mathcal{D} \cup S_1) - v(\mathbb{G}) + e(\mathcal{D} \cup S_1) \\
&= (k(\mathcal{D}) - v(\mathbb{G}) + e(\mathcal{D})) + (k(\mathcal{D} \cup S_1) - k(\mathcal{D}) + |S_1|) \\
&= n(\mathcal{D}) + n(W)
\end{aligned}$$

Let $e \in \mathcal{I}(\mathbb{Q})$. By Theorem 3, \mathbb{Q} corresponds to ρ such that e is nugatory, and by Lemma 1, $\gamma_1(\rho, e)$ is connected. Since $f(\mathbb{H}) = |\gamma_{\mathbb{H}}|$, by Lemma 1, $f(\mathcal{D} \cup e) = f(\mathcal{D}) - 1$, hence $f(\mathcal{D}) \cup S_1 = f(\mathcal{D}) - |S_1|$. Therefore, applying Lemma 4,

$$\begin{aligned}
2g(\mathcal{D} \cup S_1) &= 2k(\mathcal{D} \cup S_1) - v(\mathbb{G}) + e(\mathcal{D} \cup S_1) - f(\mathcal{D} \cup S_1) \\
&= 2k(\mathcal{D}) - v(\mathbb{G}) + (e(\mathcal{D}) + |S_1|) - (f(\mathcal{D}) - |S_1|) + 2k(\mathcal{D} \cup S_1) - 2k(\mathcal{D}) \\
&= 2g(\mathcal{D}) + 2(k(\mathcal{D} \cup S_1) - k(\mathcal{D}) + |S_1|) \\
&= 2g(\mathcal{D}) + 2n(W)
\end{aligned}$$

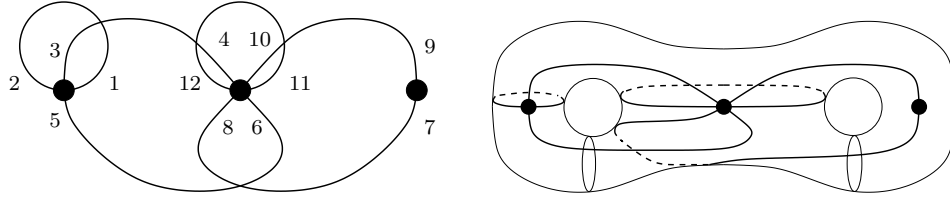
■

5 Example

In this section, we compute an example of a ribbon graph \mathbb{G} with 12 quasi-trees, whose topology varies. \mathbb{G} has three vertices and six edges, given by

$$\begin{aligned}
\sigma_0 &= (1, 3, 2, 5) (7, 9) (10, 4, 12, 8, 6, 11) \\
\sigma_1 &= (1, 2) (3, 4) (5, 6) (7, 8) (9, 10) (11, 12)
\end{aligned}$$

The ribbon graph and its surface are shown below:



In the table below, the quasi-trees are given by their coordinates in the given edge order; e.g., the vector $(0,1,1,1,0,1)$ is denoted by 011101. We also compute their chord diagrams, activities $(L, \ell$ for live; D, d for dead), numbers $\{a, b, c\} = \{n(\mathcal{D}(\mathbb{Q})), g(\mathcal{D}(\mathbb{Q})), |\mathcal{E}(\mathbb{Q})|\}$, graphs $G_{\mathbb{Q}}$, and their weights in the sum of Theorem 1. For the chord diagrams, we give the cyclic permutation of the half-edges. The types of graphs $G_{\mathbb{Q}}$ that occur in this example are as follows:

- | | |
|--------------------------------------|---|
| 1. vertex | 2. edge |
| 3. two edges with a vertex in common | 4. two edges with both vertices in common |
| 5. 2-cycle joined to an isthmus | 6. loop |
| 7. loop joined to an isthmus | |

\mathbb{Q}	$C_{\mathbb{Q}}$	Activity	$\{a, b, c\}$	$G_{\mathbb{Q}}$	Weight
001010	(1, 3, 2, 5, 11, 10, 7, 9, 4, 12, 8, 5)	$\ell d D d D d$	$\{0, 0, 1\}$	1	$(1 + Y)$
001100	(1, 3, 2, 5, 11, 10, 4, 12, 8, 9, 7, 6)	$\ell d D L d d$	$\{0, 0, 1\}$	2	$X(1 + Y)$
001111	(1, 3, 2, 5, 11, 8, 9, 4, 12, 10, 7, 6)	$\ell d D D D D$	$\{2, 1, 1\}$	1	$Y^2 Z(1 + Y)$
010010	(1, 3, 12, 8, 6, 11, 10, 7, 9, 4, 2, 5)	$\ell L d d D d$	$\{0, 0, 1\}$	2	$X(1 + Y)$
010100	(1, 3, 12, 8, 9, 7, 6, 11, 10, 4, 2, 5)	$\ell L d L d d$	$\{0, 0, 1\}$	3	$X^2(1 + Y)$
010111	(1, 3, 12, 10, 7, 6, 11, 8, 9, 4, 2, 5)	$\ell L d D D D$	$\{2, 1, 1\}$	1	$XY^2 Z(1 + Y)$
011011	(1, 3, 12, 10, 7, 9, 4, 2, 5, 11, 8, 6)	$\ell L L d D D$	$\{1, 0, 1\}$	4	$Y(1 + Y)(X + 1 + YZ)$
011101	(1, 3, 12, 10, 4, 2, 5, 11, 8, 9, 7, 6)	$\ell L L L d D$	$\{1, 0, 1\}$	5	$XY(1 + Y)(X + 1 + YZ)$
011110	(1, 3, 12, 8, 9, 4, 2, 5, 11, 10, 7, 6)	$\ell L L D D d$	$\{1, 0, 1\}$	4	$Y(1 + Y)(X + 1 + YZ)$
111010	(1, 5, 11, 10, 7, 9, 4, 2, 3, 12, 8, 6)	$L D D d D d$	$\{1, 0, 0\}$	6	$Y(1 + YZ)$
111100	(1, 5, 11, 10, 4, 2, 3, 12, 8, 9, 7, 6)	$L D D L d d$	$\{1, 0, 0\}$	7	$XY(1 + YZ)$
111111	(1, 5, 11, 8, 9, 4, 2, 3, 12, 10, 7, 6)	$L D D D D D$	$\{3, 1, 0\}$	6	$Y^3 Z(1 + YZ)$

Adding the weights in the last column, the Bollobás–Riordan–Tutte polynomial of \mathbb{G} is

$$C(\mathbb{G}) = Z^2 Y^4 + 2XZY^3 + 4ZY^3 + X^2 Y^2 + 3XY^2 + 3XZY^2 + 4ZY^2 + 2Y^2 + 2X^2 Y + 6XY + 4Y + X^2 + 2X + 1$$

As an example, let \mathbb{Q} be the eighth quasi-tree, denoted 011101. The associated partial resolution is $\rho = ****01$. $\mathcal{D}(\mathbb{Q})$ has three components, consisting of two isolated vertices and a loop. $G_{\mathbb{Q}}$ has three vertices and three edges, two connected in parallel and a second edge to the remaining vertex. The Tutte polynomial $T_{G_{\mathbb{Q}}}(x, y) = x(x + y)$. The table below gives the contributions to the state sum (1) for all $s \in [\rho]$.

For each \mathbb{H}_s , we computed the following statistics. The second column has the number of vertices, edges and faces, rank, and number of isolated vertices. The third column has the number of components, rank, nullity, and number of boundary components.

s	$\{v, e, f, r, i\}$	$\{k, r, n, bc\}$	Weight
000001	$\{1, 1, 2, 0, 2\}$	$\{3, 0, 1, 4\}$	$(X - 1)^2 Y$
000101	$\{2, 2, 2, 1, 1\}$	$\{2, 1, 1, 3\}$	$(X - 1) Y$
001001	$\{2, 2, 2, 1, 1\}$	$\{2, 1, 1, 3\}$	$(X - 1) Y$
001101	$\{3, 3, 2, 2, 0\}$	$\{1, 2, 1, 2\}$	Y
010001	$\{2, 2, 2, 1, 1\}$	$\{2, 1, 1, 3\}$	$(X - 1) Y$
010101	$\{3, 3, 2, 2, 0\}$	$\{1, 2, 1, 2\}$	Y
011001	$\{2, 3, 1, 1, 1\}$	$\{2, 1, 2, 2\}$	$(X - 1) Y^2 Z$
011101	$\{3, 4, 1, 2, 0\}$	$\{1, 2, 2, 1\}$	$Y^2 Z$
100001	$\{2, 2, 4, 0, 1\}$	$\{3, 0, 2, 5\}$	$(X - 1)^2 Y^2$
100101	$\{3, 3, 4, 1, 0\}$	$\{2, 1, 2, 4\}$	$(X - 1) Y^2$
101001	$\{2, 3, 3, 1, 1\}$	$\{2, 1, 2, 4\}$	$(X - 1) Y^2$
101101	$\{3, 4, 3, 2, 0\}$	$\{1, 2, 2, 3\}$	Y^2
110001	$\{2, 3, 3, 1, 1\}$	$\{2, 1, 2, 4\}$	$(X - 1) Y^2$
110101	$\{3, 4, 3, 2, 0\}$	$\{1, 2, 2, 3\}$	Y^2
111001	$\{2, 4, 2, 1, 1\}$	$\{2, 1, 3, 3\}$	$(X - 1) Y^3 Z$
111101	$\{3, 5, 2, 2, 0\}$	$\{1, 2, 3, 2\}$	$Y^3 Z$

The contribution to the state sum (1) from the interval $[\rho]$ is the sum of the weights in the last column. In accordance with Theorem 1, this sum equals

$$Y^{n(\mathcal{D}(\mathbb{Q}))} Z^{g(\mathcal{D}(\mathbb{Q}))} (1 + Y)^{|\mathcal{E}(\mathbb{Q})|} T_{G_{\mathbb{Q}}}(X, 1 + YZ) = Y(Y + 1)X(X + 1 + YZ)$$

References

- [1] B. Bollobás and O. Riordan. A polynomial invariant of graphs on orientable surfaces. *Proc. London Math. Soc. (3)*, 83(3):513–531, 2001.
- [2] B. Bollobás and O. Riordan. A polynomial of graphs on surfaces. *Math. Ann.*, 323(1):81–96, 2002.
- [3] O. Dasbach, D. Futer, E. Kalfagianni, X.-S. Lin, and N. Stoltzfus. Alternating sum formulae for the determinant and other link invariants. arXiv:math.GT/0611025.
- [4] W. Tutte. A contribution to the theory of chromatic polynomials. *Canadian J. Math.*, 6:80–91, 1954.